TOPOLOGICAL ENTROPY FOR THE CANONICAL ENDOMORPHISM OF CUNTZ-KRIEGER ALGEBRAS

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Let Σ be a finite set and let $A = \big(A(i,j)\big)_{i,j\in\Sigma}$ such that $A(i,j) \in \{0,1\}$ and all rows and columns of A are non-zero. The Cuntz-Krieger algebra $\mathcal{O}_A$ is the universal C^*-algebra generated by partial isometries $S_i \neq 0$, $i \in \Sigma$, with the property that their support projections $Q_i = S_i^* S_i$ and $P_i = S_i S_i^*$ satisfy the relations

$$P_i P_j = \delta_{ij} P_i, \qquad Q_i = \sum_{j\in\Sigma} A(i,j) P_j, \qquad i,j \in \Sigma.$$

The aim of this note is to compute the topological entropy of the canonical "endomorphism" $\phi_A : \mathcal{O}_A \to \mathcal{O}_A$, which is the ucp (unital completely positive) map defined as

$$\phi_A(X) = \sum_{j\in\Sigma} S_j X S_j^*, \qquad X \in \mathcal{O}_A.$$

The map ϕ_A plays a crucial rôle in the study of $\mathcal{O}_A$ ([6]). It invariates the AF-part $\mathcal{F}_A$ of $\mathcal{O}_A$ and the abelian subalgebra $\mathcal{D}_A$ generated by $\phi_A^k(P_i)$, $i \in \Sigma$, $k \in \mathbf{N}$. The restriction $\phi_A|_{\mathcal{D}_A}$ is an isometric endomorphism of $\mathcal{D}_A$. Actually $\mathcal{D}_A$ identifies with $C(X_A)$, the commutative C^*-algebra of continuous functions on the compact space

$$X_A = \big\{ (x_k)_{k\in\mathbf{N}}\,;\, x_k \in \Sigma,\ A(x_k, x_{k+1}) = 1 \big\}$$

and ϕ_A is the endomorphism induced on $C(X_A)$ by the one-sided subshift of finite type σ_A (see [6]) defined by

$$(\sigma_A x)_k = x_{k+1}, \qquad x = (x_k)_{k\in\mathbf{N}} \in X_A.$$

Therefore ϕ_A can be regarded as a non-commutative generalization of the one-sided subshift of finite type associated with the matrix A and the computation of its dynamical entropies is of some interest (see [4, Page 691]).

When $A(i,i) = 1$, $i \in \Sigma$, one gets the Cuntz algebra $\mathcal{O}_N$ where N is the cardinality of Σ (see [5]). In this case $\phi_N = \sum_{j=1}^N S_j \cdot S_j^*$ is a genuine endomorphism (i.e. $\phi_N(XY) = \phi_N(X)\phi_N(Y)$, $X, Y \in \mathcal{O}_N$) which invariates the AF-part $\mathcal{F}_N = \bigotimes_1^\infty M_N(\mathbf{C})$ of $\mathcal{O}_N$ and $\phi_N|_{\mathcal{F}_N}$ coincides with the noncommutative Bernoulli shift $\phi_N(X) = 1 \otimes X$, $X \in \mathcal{F}_N$. Furthermore, $\phi_N|_{\mathcal{D}_N}$ is the classical one-sided Bernoulli shift.

D. Voiculescu has introduced in [9] a notion of topological entropy for noncommutative dynamical systems (A, α), where A is a unital nuclear C^*-algebra and α an automorphism (or endomorphism) of A which extends the classical commutative topological entropy. In the noncommutative framework partitions of unity are being replaced by ucp map ([4],[9]). As pointed out by N. Brown (see [1]), Voiculescu's definition carries on, with slight modifications, to the larger class of (not necessarily unital) exact C^*-algebras.

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M. Choda has computed in [2] the topological entropy $\mathrm{ht}(\phi_N)$ of the canonical endomorphism ϕ_N on $\mathcal{O}_N$, proving $\mathrm{ht}(\phi_N) = \log N$. The equality $ht\big(\phi_A|_{\mathcal{F}_A}\big) = \log r(A)$ has been proved in [7]. In this note we extend these results and compute, under a suitable definition for the topological entropy of a cp map, the topological entropy $\mathrm{ht}(\phi_A)$, proving

Theorem. *If A is irreducible and not a permutation matrix, then*

$$\mathrm{ht}(\phi_A) = \log r(A).$$

Here $r(A)$ denotes the spectral radius of A, which coincides by Perron-Frobenius with the largest (positive) eigenvalue of A.

One can associate to any matrix $A = \big(A(i,j)\big)_{i,j\in\Sigma}$ with $A(i,j) \in \mathbf{Z}^+$ its dual matrix $A' = \big(A'(r,s)\big)_{r,s\in\Sigma'}$ with $A'(r,s) \in \{0,1\}$ and define $\mathcal{O}_A$ as $\mathcal{O}_{A'}$ (see [6]). Since $A = ST$ and $A' = TS$ for some matrices S and T, one has $r(A') = r(A)$. Hence the topological entropy of the canonical endomorphism $\phi_{A'}$ on $\mathcal{O}_{A'} = \mathcal{O}_A$ equals $r(A)$.

1. Proof of the main result

We first recall some basic definitions from [1] and [9]. In the sequel $\mathcal{A}$ will be an exact C^*-algebra and $\mathcal{P}f(\mathcal{A})$ will denote the set of finite subsets of $\mathcal{A}$. For any faithful $*$-representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ one denotes by $CPA(\pi, \mathcal{A})$ the set of triples $(\phi, \psi, \mathcal{B})$, where $\mathcal{B}$ is a finite-dimensional C^*-algebra and $\phi : \mathcal{A} \to \mathcal{B}$, $\psi : \mathcal{B} \to \mathcal{B}(\mathcal{H})$ are cp maps. One also considers for any $\omega \in \mathcal{P}f(\mathcal{A})$ the completely positive δ-rank

$$rcp(\pi, \omega; \delta) = \inf\big\{\mathrm{rank}\,\mathcal{B}\,;\ (\phi, \psi, \mathcal{B}) \in CPA(\pi, \mathcal{A}),\ \|\psi\phi(a) - \pi(a)\| < \delta,\ a \in \omega\big\}. \tag{1}$$

By an important result of E. Kirchberg and S. Wassermann exact C^*-algebras are nuclearly embeddable (see [10]). Hence, there exists π faithful such that for all $\omega \in \mathcal{P}f(\mathcal{A})$ and $\delta > 0$, there is $(\phi, \psi, \mathcal{B}) \in CPA(\pi, \mathcal{A})$ with $\big\|\psi\phi(a) - \pi(a)\big\| < \delta$, $a \in \omega$. As noticed in [1], $rcp(\pi, \omega; \delta)$ is independent on the choice of π.

Assume also that $\Phi : \mathcal{A} \to \mathcal{A}$ is a cp map, let $\mathcal{A} \hookrightarrow \mathcal{B}(\mathcal{H})$ and define

$$ht(\Phi, \omega; \delta) = \limsup_n n^{-1} \log rcp\big(\omega \cup \Phi(\omega) \cup \cdots \cup \Phi^{n-1}(\omega)\,;\, \delta\big),$$

$$ht(\Phi, \omega) = \sup_{\delta>0} ht(\Phi, \omega; \delta), \qquad ht(\Phi) = \sup_{\omega\in\mathcal{P}f(\mathcal{A})} ht(\alpha, \omega).$$

As in the case of automorphisms or endomorphisms, $ht(\Phi)$ enjoys some basic properties which are collected in the next proposition. Proofs are similar to the corresponding ones from [1] and [9].

Proposition 1. *(i)* (Monotonicity) *Let $\mathcal{A}_0$ be a subalgebra of $\mathcal{A}$ such that $\Phi(\mathcal{A}_0) \subset \mathcal{A}_0$. Then*

$$ht\big(\Phi|_{\mathcal{A}_0}\big) \le ht(\Phi).$$

(ii) (Kolmogorov-Sinai type property) *Let $\omega_j \in \mathcal{P}f(\mathcal{A})$ such that $\omega_0 \subset \omega_1 \subset \dots$ and the linear span of $\bigcup_{j,k\in\mathbf{N}} \Phi^k(\omega_j)$ is norm dense in $\mathcal{A}$. Then*

$$ht(\Phi) = \sup_j ht(\Phi, \omega_j).$$

(iii) (Invariance to outer conjugacy) *For any $\theta \in \mathrm{Aut}(\mathcal{A})$ one has*

$$ht(\theta\Phi\theta^{-1}) = ht(\Phi).$$

(iv) *For any* $k \in \mathbf{N}$, $\omega \in \mathcal{P}f(\mathcal{A})$ *and* $\delta > 0$ *one has*

$$ht(\Phi^k, \omega; \delta) \leq k\, ht(\Phi, \omega; \delta).$$

(v) *For any cp maps* $\Phi_j : \mathcal{A}_j \to \mathcal{A}_j$, $j = 1, 2$, *one has*

$$\max\big(ht(\Phi_1), ht(\Phi_2)\big) \leq ht(\Phi_1 \otimes \Phi_2) \leq ht(\Phi_1) + ht(\Phi_2).$$

Next we turn to the Cuntz-Krieger C^*-algebra $\mathcal{O}_A$ associated with an irreducible, non-permutation matrix A with entries in $\{0, 1\}$. For any k-tuple $\mu = (\mu_1, \ldots, \mu_k)$, $\mu_j \in \Sigma$, we denote $o(\mu) = \mu_1$, $t(\mu) = \mu_k$, $S_\mu = S_{\mu_1} \ldots S_{\mu_k}$, $S_e = I$, $o(e) = t(e) = I$ (e denotes the empty word) and $Q_\mu = S_\mu^* S_\mu$. For $\mu = (\mu_1, \ldots, \mu_k), \nu = (\nu_1, \ldots, \nu_l)$, $\mu_i, \nu_j \in \Sigma$ we denote $\mu\nu = (\mu_1, \ldots, \mu_k, \nu_1, \ldots, \nu_l)$. The number of elements of a finite set F is denoted by $\#F$. We set $A(\mu) = 1$ for $k = 1$ and

$$A(\mu) = A(\mu_1, \mu_2)A(\mu_2, \mu_3) \ldots A(\mu_{k-1}, \mu_k) \qquad \text{for } k \geq 2.$$

Then, for μ, ν with $|\mu| = |\nu|$ one has

$$S_\mu^* S_\nu = \delta_{\mu\nu} Q_\mu = \delta_{\mu\nu} A(\mu) Q_{t(\mu)},$$

$$Q_\eta S_\alpha = A\big(\eta o(\alpha)\big) S_\alpha \qquad \text{for } |\alpha| \geq 1.$$

In particular $S_\mu \neq 0$, $|\mu| \geq 1$, is equivalent to $A(\mu) \neq 0$. It is clear that the number of elements of

$$L(k) = \big\{\mu\,;\, |\mu| = k,\ S_\mu \neq 0\big\}$$

equals

$$\begin{aligned} w(k) &= \#\big\{(i_1, \ldots, i_k)\,;\, i_j \in \Sigma,\ A(i_1, i_2)A(i_2, i_3) \ldots A(i_{k-1}, i_k) = 1\big\} \\ &= \sum_{i_1, \ldots, i_k \in \Sigma} A(i_1, i_2)A(i_2, i_3) \ldots A(i_{k-1}, i_k) = \sum_{i,j \in \Sigma} A^{k-1}(i, j) \\ &= \langle A^{k-1} e, e \rangle, \qquad \text{where } e = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}. \end{aligned} \tag{2}$$

Note also that if $r(A)$ denotes the spectral radius of A, then

$$\|A^{k-1}\| \leq \sum_{i,j \in \Sigma} A^{k-1}(i, j) = w(k) \leq \|A^{k-1}\| \cdot \|e\|_2^2 = \#\Sigma \cdot \|A^{k-1}\|,$$

which provides

$$\lim_k k^{-1} \log w(k) = \lim_k k^{-1} \log \|A^{k-1}\| = \log r(A). \tag{3}$$

We consider now a certain embedding of the Cuntz-Krieger algebra $\mathcal{O}_A$ into $M_{w(m)}(\mathbf{C}) \otimes \mathcal{O}_A$. For each $m \geq 1$ we index the canonical matrix unit of $M_{w(m)}(\mathbf{C})$ as $\{e_{\mu\nu}\}_{\mu,\nu \in L(m)}$ and define a map $\rho_m : \mathcal{O}_A \to M_{w(m)}(\mathbf{C}) \otimes \mathcal{O}_A$ by

$$\rho_m(X) = \sum_{\mu,\nu \in L(m)} e_{\mu\nu} \otimes S_\mu^* X S_\nu. \tag{4}$$

Since $\sum_{\mu \in L(m)} S_\mu S_\mu^* = \sum_{|\mu| = m} S_\mu S_\mu^* = I$, it is easily seen that ρ_m is a $*$-morphism. Moreover, since $\mathcal{O}_A$ is simple, it follows that $\rho_m : \mathcal{O}_A \to \rho_m(\mathcal{O}_A) \subset M_{w(m)}(\mathbf{C}) \otimes \mathcal{O}_A$ is a $*$-isomorphism. The map ρ_m is not unital in general. We only have $\rho_m(1) = \sum_{\mu \in L(m)} e_{\mu\mu} \otimes Q_\mu = \sum_{\mu \in L(m)} e_{\mu\mu} \otimes Q_{t(\mu)}$.

However, for Cuntz algebras this map is unital, multiplicative and onto, providing an explicit isomorphism between $\mathcal{O}_N$ and $M_{N^m}(\mathbf{C})\otimes\mathcal{O}_N$. To see that ρ_m is onto note that for $|\mu_0|=|\nu_0|=m$ we have $\rho_m(S_{\mu_0}S_{\nu_0}^*)=e_{\mu_0\nu_0}\otimes I$ and $\rho_m(S_{\mu_0}S_{\nu_0}S_{\mu_0}^*)=e_{\mu_0\mu_0}\otimes S_{\nu_0}$. For $m=1$ and $N=2$ this map was used by M. D. Choi (see [3]) to prove the isomorphism between $M_2(\mathbf{C})\otimes\mathcal{O}_2$ and $\mathcal{O}_2$.

We return to the general case and note that for all $l\geq 1$

$$\phi_A^l(X)=\sum_{|\eta|=l}S_\eta X S_\eta^*=\sum_{\eta\in L(l)}S_\eta X S_\eta^*,\qquad X\in\mathcal{O}_A. \tag{5}$$

Lemma 2. *Let $n\geq 1$ and assume that $|\beta|\leq|\alpha|\leq n_0$ and $m\geq n+n_0$. Then, for all $i\in\Sigma$ and all $l\in\{0,1,\dots,n-1\}$ one has*

$$\rho_m\phi_A^l(S_\alpha P_i S_\beta^*)=\begin{cases}\displaystyle\sum_{|\mu|=|\alpha|-|\beta|}X(\mu)\otimes S_\mu & \text{if } |\beta|<|\alpha|,\\ \displaystyle\sum_{j\in\Sigma}X_j\otimes Q_j & \text{if } |\beta|=|\alpha|,\end{cases}$$

for some partial isometries $X(\mu)=X\big(|\alpha|,|\beta|,i,l,\mu\big)$ and respectively $X_j=X\big(|\alpha|,i,l,j\big)$.

Proof. From (5) and (4) we get

$$\begin{aligned}\rho_m\phi_A^l(S_\alpha P_i S_\beta^*)&=\sum_{\eta\in L(l)}\rho_m(S_{\eta\alpha}P_iS_{\eta\beta}^*)=\sum_{\eta\in L(l)}\sum_{\mu,\nu\in L(m)}e_{\mu\nu}\otimes S_\mu^*S_\eta S_\alpha P_i S_\beta^* S_\eta^* S_\nu\\ (\textit{with } \mu=\eta\mu',\ \nu=\eta\nu')\quad&=\sum_{|\eta|=l}\sum_{\substack{|\mu'|=|\nu'|=m-l\\ \eta\mu',\eta\nu'\in L(m)}}e_{\eta\mu',\eta\nu'}\otimes S_{\mu'}^*Q_\eta S_\alpha P_i S_\beta^* Q_\eta S_{\nu'}.\end{aligned}\tag{6}$$

For $|\beta|=|\alpha|\geq 1$ this yields

$$\begin{aligned}\rho_m\phi_A^l(S_\alpha P_i S_\beta^*)&=\sum_{|\eta|=l}\sum_{\substack{|\mu'|=|\nu'|=m-l\\ \eta\mu',\eta\nu'\in L(m)}}A\big(\eta o(\alpha)\big)A\big(\eta o(\beta)\big)e_{\eta\mu',\eta\nu'}\otimes S_{\mu'}^*S_\alpha P_i S_\beta^* S_{\nu'}\\ (\textit{with } \mu'=\alpha\mu'',\ \nu'=\beta\nu'')\quad&=\sum_{|\eta|=l}\sum_{\substack{|\mu''|=|\nu''|=m-l-|\alpha|\\ \eta\alpha\mu'',\eta\beta\nu''\in L(m)}}A\big(\eta o(\alpha)\big)A\big(\eta o(\beta)\big)e_{\eta\alpha\mu'',\eta\beta\nu''}\otimes S_{\mu''}^*Q_\alpha P_i Q_\beta S_{\nu''}\\ &=\sum_{|\eta|=l}\sum_{\substack{|\mu''|=|\nu''|=m-l-|\alpha|\\ \eta\alpha\mu'',\eta\beta\nu''\in L(m)}}A\big(\eta\alpha o(\mu'')\big)A\big(\eta\beta o(\nu'')\big)e_{\eta\alpha\mu'',\eta\beta\nu''}\otimes S_{\mu''}^*P_i S_{\nu''}\\ &=\sum_{|\eta|=l}\sum_{\substack{|\mu''|=|\nu''|=m-l-|\alpha|\\ o(\mu'')=o(\nu'')=i\\ \eta\alpha\mu'',\eta\beta\nu''\in L(m)}}A(\eta\alpha i)A(\eta\beta i)e_{\eta\alpha\mu'',\eta\beta\nu''}\otimes S_{\mu''}^*S_{\nu''}\\ (\textit{with } \nu''=\mu'')\quad&=\sum_{|\eta|=l}\sum_{\substack{|\mu''|=m-l-|\alpha|,\,o(\mu'')=i\\ \eta\alpha\mu'',\eta\beta\mu''\in L(m)}}A(\eta\alpha i)A(\eta\beta i)A(\mu'')e_{\eta\alpha\mu'',\eta\beta\mu''}\otimes Q_{t(\mu'')}\\ &=\sum_{|\eta|=l}\sum_{\substack{|\mu''|=m-l-|\alpha|,\,o(\mu'')=i\\ \eta\alpha\mu'',\eta\beta\mu''\in L(m)}}e_{\eta\alpha\mu'',\eta\beta\mu''}\otimes Q_{t(\mu'')}\\ &=\sum_{j\in\Sigma}X_j\otimes Q_j,\end{aligned}$$

where

$$X_j = \sum_{|\eta|=l} \sum_{\substack{|\mu''|=m-l-|\alpha| \\ o(\mu'')=i,\, t(\mu'')=j \\ \eta\alpha\mu'',\eta\beta\mu''\in L(m)}} e_{\eta\alpha\mu'',\eta\beta\mu''}$$

are partial isometries for all $j \in \Sigma$.

For $\beta = \alpha = e$ a similar computation yields $\rho_m\phi_A^l(P_i) = \sum_{j\in\Sigma} X_j \otimes Q_j$, with

$$X_j = \sum_{|\eta|=l} \sum_{\substack{|\mu'|=m-l,\, o(\mu')=i,\, t(\mu')=j \\ \eta\mu'\in L(m)}} e_{\eta\mu',\eta\mu'}.$$

For $1 \leq |\beta| < |\alpha|$ equality (6) yields

$$\begin{aligned}
\rho_m\phi_A^l(S_\alpha P_i S_\beta^*) &= \sum_{|\eta|=l} \sum_{\substack{|\mu'|=|\nu'|=m-l \\ \eta\mu',\eta\nu'\in L(m)}} A\big(\eta o(\alpha)\big) A\big(\eta o(\beta)\big) e_{\eta\mu',\eta\nu'} \otimes S_{\mu'}^* S_\alpha P_i S_\beta^* S_{\nu'} \\
(\mathit{with}\ \mu' = \alpha\mu'',\ \nu' = \beta\nu'') &= \sum_{|\eta|=l} \sum_{\substack{|\mu''|=m-l-|\alpha| \\ |\nu''|=m-l-|\beta| \\ \eta\alpha\mu'',\eta\beta\nu''\in L(m)}} A\big(\eta o(\alpha)\big) A\big(\eta o(\beta)\big) e_{\eta\alpha\mu'',\eta\beta\nu''} \otimes S_{\mu''}^* Q_\alpha P_i Q_\beta S_{\nu''} \\
&= \sum_{|\eta|=l} \sum_{\substack{|\mu''|=m-l-|\alpha| \\ |\nu''|=m-l-|\beta| \\ \eta\alpha\mu'',\eta\beta\nu''\in L(m)}} A\big(\eta\alpha o(\mu'')\big) A\big(\eta\beta o(\nu'')\big) e_{\eta\alpha\mu'',\eta\beta\nu''} \otimes S_{\mu''}^* P_i S_{\nu''} \\
&= \sum_{|\eta|=l} \sum_{\substack{|\mu''|=m-l-|\alpha|,\, o(\mu'')=i \\ |\nu''|=m-l-|\beta|,\, o(\nu'')=i \\ \eta\alpha\mu'',\eta\beta\nu''\in L(m)}} A(\eta\alpha i) A(\eta\beta i) e_{\eta\alpha\mu'',\eta\beta\nu''} \otimes S_{\mu''}^* S_{\nu''} \\
(\mathit{with}\ \nu'' = \mu''\mu) &= \sum_{|\eta|=l} \sum_{\substack{|\mu''|=m-l-|\alpha|,\, o(\mu'')=i \\ |\mu|=|\alpha|-|\beta| \\ \eta\alpha\mu'',\eta\beta\mu''\mu\in L(m)}} A\big(\eta\alpha\mu'' o(\mu)\big) A(\eta\beta i) e_{\eta\alpha\mu'',\eta\beta\mu''\mu} \otimes S_\mu \\
&= \sum_{|\eta|=l} \sum_{\substack{|\mu''|=m-l-|\alpha|,\, o(\mu'')=i \\ |\mu|=|\alpha|-|\beta| \\ \eta\alpha\mu'',\eta\beta\mu''\mu\in L(m)}} e_{\eta\alpha\mu'',\eta\beta\mu''\mu} \otimes S_\mu \\
&= \sum_{|\mu|=|\alpha|-|\beta|} X(\mu) \otimes S_\mu,
\end{aligned}$$

where

$$X(\mu) = \sum_{|\eta|=l} \sum_{\substack{|\mu''|=m-l-|\alpha|,\, o(\mu'')=i \\ \eta\alpha\mu'',\eta\beta\mu''\mu\in L(m)}} e_{\eta\alpha\mu'',\eta\beta\mu''\mu}$$

are partial isometries for all $\mu \in L(|\alpha| - |\beta|)$. One plainly checks that for $\beta = e$, $|\alpha| \geq 1$, the formula $\rho_m\phi_A^l(S_\alpha P_i) = \sum_{|\mu|=|\alpha|} X(\mu) \otimes S_\mu$ holds for the $X(\mu)$ above which corresponds to $\beta = e$. $\square$

For any $k \geq 1$ we put

$$\omega(k) = \big\{ S_\alpha P_i S_\beta^*\,;\, |\beta| \leq |\alpha| \leq k,\ i \in \Sigma \big\}.$$

Proposition 3. *For all $n_0 \geq 1$ and $\delta > 0$ one has*

$$\limsup_n n^{-1} \log rcp\left(\omega(n_0) \cup \phi_A\big(\omega(n_0)\big) \cup \cdots \cup \phi_A^{n-1}\big(\omega(n_0)\big)\,;\, \delta\right) \leq \log r(A).$$

Proof. For $n \geq 1$ we let $m = m(n) = n + n_0$. Since $\mathcal{O}_A$ is nuclear, there exists $\big(\phi_0, \psi_0, M_{m_0}(\mathbf{C})\big) \in CPA(id_A, \mathcal{O}_A)$, that is

$$\begin{tikzcd} \mathcal{O}_A \arrow[rr, "id_{\mathcal{O}_A}"] \arrow[dr, "\phi_0"'] & & \mathcal{O}_A \\ & M_{m_0}(\mathbf{C}) \arrow[ur, "\psi_0"'] & \end{tikzcd}$$

such that

$$\big\|\psi_0\phi_0(Q_j) - Q_j\big\| + \big\|\psi_0\phi_0(S_\gamma) - S_\gamma\big\| < \frac{\delta}{\max\big(\#\Sigma, w(n_0)\big)} \qquad \text{for all } \gamma \in L(n_0) \text{ and } j \in \Sigma. \tag{7}$$

Consider $\mathcal{B} = M_{w(m)}(\mathbf{C}) \otimes M_{m_0}(\mathbf{C})$ and let $\mathcal{H}$ be a Hilbert space on which $\mathcal{O}_A$ acts faithfully. The $*$-isomorphism $\rho_m^{-1} : \rho_m(\mathcal{O}_A) \to \mathcal{O}_A$ extends to a cp map $\Psi_m : M_{w(m)}(\mathbf{C}) \otimes \mathcal{O}_A \to \mathcal{B}(\mathcal{H})$ with $\|\Psi_m\| = 1$. We consider the cp maps $\phi = (id \otimes \phi_0)\rho_m : \mathcal{O}_A \to \mathcal{B}$ and $\psi = \Psi_m(id \otimes \psi_0) : \mathcal{B} \to \mathcal{B}(\mathcal{H})$; see the following diagram

$$\begin{tikzcd} \mathcal{O}_A \arrow[r, "\rho_m"] \arrow[ddrr, "\phi"'] \arrow[dr, phantom, "="] & \rho_m(\mathcal{O}_A) \arrow[rr, "id_{\rho_m(\mathcal{O}_A)}"] \arrow[ddr, "id\otimes\phi_0"] & & \rho_m(\mathcal{O}_A) \arrow[r, "\rho_m^{-1}"] \arrow[d, hook] \arrow[dr, phantom, "="] & \mathcal{B}(\mathcal{H}) \\ & & & M_{w(m)}(\mathbf{C}) \otimes \mathcal{O}_A \arrow[ur, dashed, "\Psi_m"'] \arrow[d, phantom, "="] & \\ & & \mathcal{B} = M_{w(m)}(\mathbf{C}) \otimes M_{m_0}(\mathbf{C}) \arrow[ur, "id\otimes\psi_0"'] \arrow[uurr, bend right, "\psi"'] & & \end{tikzcd}$$

Let $a = S_\alpha P_i S_\beta^* \in \omega(n_0)$. By the previous lemma there exist partial isometries $X(\mu) = X(a, l, \mu)$ if $|\beta| < |\alpha|$ and $X_j = X(a, l, j)$ if $|\alpha| = |\beta|$ such that

$$\rho_m \phi_A^l(a) = \begin{cases} \displaystyle\sum_{|\mu| = |\alpha| - |\beta|} X(\mu) \otimes S_\mu & \text{for } |\beta| < |\alpha|, \\ \displaystyle\sum_{j \in \Sigma} X_j \otimes Q_j & \text{for } |\beta| = |\alpha|. \end{cases} \tag{8}$$

From (8) and (7) we gather

$$\begin{aligned}
&\big\|\psi\phi\big(\phi_A^l(a)\big) - \phi_A^l(a)\big\| = \big\|\Psi_m(id\otimes\psi_0\phi_0)\big(\rho_m\phi_A^l(a)\big) - \phi_A^l(a)\big\| \\
&\quad = \big\|\Psi_m(id\otimes\psi_0\phi_0)\big(\rho_m\phi_A^l(a)\big) - \Psi_m\big(\rho_m\phi_A^l(a)\big)\big\| \le \big\|(id\otimes\psi_0\phi_0)\big(\rho_m\phi_A^l(a)\big) - \rho_m\phi_A^l(a)\big\| \\
&\quad = \begin{cases} \Big\| \displaystyle\sum_{|\mu|=|\alpha|-|\beta|} X(\mu)\otimes\big(\psi_0\phi_0(S_\mu)-S_\mu\big)\Big\| & \text{for } |\beta|<|\alpha|, \\ \Big\|\displaystyle\sum_{j\in\Sigma} X_j\otimes\big(\psi_0\phi_0(Q_j)-Q_j\big)\Big\| & \text{for } |\beta|=|\alpha|, \end{cases} \\
&\quad < \max\big(\#\Sigma, w(n_0)\big)\cdot\frac{\delta}{\max\big(\#\Sigma, w(n_0)\big)} = \delta.
\end{aligned}$$

Therefore

$$rcp\left(\omega(n_0)\cup\phi_A\big(\omega(n_0)\big)\cup\cdots\cup\phi_A^{n-1}\big(\omega(n_0)\big)\,;\,\delta\right) \le m_0 w(m) = m_0 w(n+n_0),$$

which we combine with (3) to get

$$\limsup_n n^{-1}\log\left(\omega(n_0)\cup\cdots\cup\phi_A^{n-1}\big(\omega(n_0)\big)\,;\,\delta\right) \le \limsup_n n^{-1}\log w(n) = \log r(A). \qquad \square$$

Proof of the main result. Since $\omega_n = \omega(n)\cup\omega(n)^*$ is an increasing sequence of finite subsets of $\mathcal{O}_A$ and $\mathrm{span}\bigcup_n\omega_n$ is dense in the uniform norm in $\mathcal{O}_A$, Proposition 1 (ii) and Proposition 3 provide

$$ht(\phi_A) \le \log r(A). \tag{9}$$

For the opposite inequality, denote $\theta_A = \phi_A|_{\mathcal{D}_A = C(X_A)}$. By Proposition 1 (i) $ht(\phi_A)\ge ht(\theta_A)$. Within the framework of [4], let σ be a probability measure on X_A such that $\sigma\theta_A = \sigma$. For any finite-dimensional algebra M and any ucp map $\gamma : M\to C(X_A)$, Proposition III.6 in [4] provides $H_\sigma(\gamma,\theta_A\gamma,\dots,\theta_A^{n+m-1}\gamma)\le H_\sigma(\gamma,\theta_A\gamma,\dots,\theta_A^{n-1}\gamma) + H_\sigma(\gamma,\theta_A\gamma,\dots,\theta_A^{n-1}\gamma)$ for all $m,n\ge 1$, hence

$$h_{\sigma,\theta_A}(\gamma) = \lim_n n^{-1} H_\sigma(\gamma,\theta_A\gamma,\dots,\theta_A^{n-1}\gamma)$$

exists. Let $h_\sigma(\theta_A)$ be the supremum of $h_{\sigma,\theta_A}(\gamma)$ over all such M and γ. Arguing as in [9, Prop.4.8] one proves that for any $\gamma : M\to C(X_A)$ as above and any $\varepsilon>0$, there exist $\omega\in\mathcal{P}f\big(C(X_A)\big)$ and $\delta>0$ such that

$$H_\sigma(\gamma,\theta_A\gamma,\dots,\theta_A^{n-1}\gamma)\le n\varepsilon + \log rcp\left(\omega\cup\theta_A(\omega)\cup\cdots\cup\theta_A^{n-1}(\omega);\delta\right),$$

hence

$$h_{\sigma,\theta_A}(\gamma)\le ht(\theta_A). \tag{10}$$

If $\mathcal{P}$ is a finite partition into time-zero cylinder sets, $\mathcal{P}_n = \mathcal{P}\vee\sigma_A^{-1}\mathcal{P}\vee\cdots\vee\sigma_A^{-(n-1)}\mathcal{P}$, $\mathcal{C}$ is the abelian finite-dimensional C^*-algebra generated by $\mathcal{P}_n$ and $\gamma = i_{\mathcal{C}}$ the natural inclusion of $\mathcal{C}$ into $C(X_A)$, then

$$-\sum_{E\in\mathcal{P}_n}\sigma(\chi_E)\log\sigma(\chi_E) = S\big(\sigma|_{\mathcal{P}_n}\big) = H_\sigma(\gamma,\theta_A\gamma,\dots,\theta_A^{n-1}\gamma), \tag{11}$$

the last equality following from [4, Remark III.5.2]. From (10) and (11) it follows that the classical measurable entropy $H_\sigma(\sigma_A)$ is $\le ht(\theta_A)$. In the case when σ is the probability measure defined by a

probability eigenvector of A it is well-known (see e.g. [8]) that $H_\sigma(\sigma_A) = \log r(A)$. Hence one has $ht(\phi_A) \geq ht(\theta_A) \geq \log r(A)$, which completes the proof. $\square$

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